# A NOTE ON FLAME STABILITY THEORY

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Inzhenerno-Fizicheskii Zhurnal, Vol. 11, No. 4, pp. 472–478, 1966

UDC 536.46

The stability of normal combustion of a gas mixture with respect to small hydrodynamic perturbations is examined. An explanation is given of the physically contradictory conclusion obtained in [1] regarding the destabilizing influence of viscosity. With the aid of the author's inverse feedback condition [2] for a flame zone of finite thickness stability is investigated with allowance for viscous forces. The results show good agreement with well-known experiments [3, 4] on the instability of spherical flames.

The problem of the stability of normal combustion of a viscous gas mixture with respect to small hydrodynamic perturbations is investigated below in the following scheme. A plane flame, parallel to the y axis and propagating in the direction of the negative x axis, has a finite dimension L along this axis. To the left and right of the zone L there are, respectively flows of original mixture and of combustion products, which are assumed to be constant in the undisturbed state. Within the flame zone there is a continuous transition from the parameters of the unburnt gas to those of the burnt gas. The stability of the above combustion system has previously been analyzed by Einbinder [1], who represented solutions of the linearized Navier-Stokes equations of motion of a viscous fluid and of the continuity equation in the form $f(x) = \exp(ihy + \delta t)$. Because of the constancy of the parameters for the regions of the original mixture and the combustion products, the function $f(x)$ is represented there as an exponential, while, within the flame zone, $f(x)$ finally satisfies a fourth-order differential equation which contains the viscosity $\eta$ as an important parameter. On the front and rear boundaries of the flame zone four boundary conditions are formulated for this equation, in the form of a smooth junction of the perturbed state of this zone with the burnt and unburnt gas regions. Later on, in §6 [1], Einbinder makes a transition to a discontinuous flame front, letting the dimensionless thickness $\xi_{\text{flame}} = hL$ go to zero, and retaining the viscosity effect $\eta$ everywhere in the formulas. As a result an equation is obtained for determining the eigenvalue $\delta$, from which follows the conclusion regarding the destabilizing influence of the viscosity. The reason for this result, contradicting as it does the physical picture, is Einbinder's inconsistent passage to the limit ($\xi = 0$) in the solutions for perturbations of a flame of finite thickness. Indeed, as noted in §2 by Einbinder himself, the "viscous" thickness ($\xi$ is density, and velocity), or, in dimensionless form, $\xi_{\text{flame}} = hL \sim \eta$. Therefore, the correct passage to the limit $\xi = 0$ (L = 0) in the solutions for perturbations of the flame must be accompanied by $\eta \to 0$, i. e., by discarding the viscosity. Einbinder, by taking L = 0 ($\xi = 0$), but $\eta \neq 0$, thus took only partial account of the linear term in the expansion of the solution for perturbations of the flame with respect to viscosity $\eta$, which invalidates the physically inconsistent result that he obtained. Moreover, we have a remark apropos the transition to a discontinuous front: the behavior of a strong discontinuity is controlled, in general, not by the differential equations, but by the physical laws of conservation of mass flow and momentum flow across the discontinuity. With the objective of explaining the viscous effect, we undertake below to investigate the stability of a normal flame flame (on the model described above) on the basis of the the inverse feedback equation previously described in [2], bringing in also the general theorems of the mechanics of continua [5] in order to join the burnt and unburnt gas regions across the flame zone: these theorems are the momentum and conservation of mass theorems.

In a coordinate system fixed relative to a flame propagating in the undisturbed state, let the chemical reaction of combustion be concluded at time t on the y axis, in such a way that the flame zone occupies the interval $-L \leq x \leq 0$, and the gas flow proceeds in the direction of the positive x axis. We denote the parameters p, $\rho$, v, T, $\eta$, $\chi$ of the flow with subscripts 1, 2, and 3 relating to the regions: original mixture (x > 0), combustion products (x > 0), and flame ($-L < x < 0$). For gases, the dependence of viscosity on temperature ordinarily [5] has the form

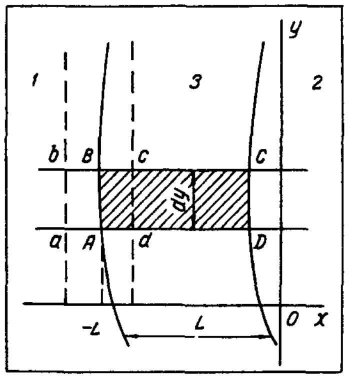
$$\eta T^{-m} = \text{const}\ (0.5 \leqslant m \leqslant 1). \tag{1}$$

The small velocity of normal combustion permits us to assume that the medium incompressible. Because of the strong dependence of the rate of the chemical reaction on temperature, the latter takes place in a narrow range of temperature, such that the reaction zone comprises a relatively small part of the total width L of the flame. Also, as noted in [2], the lack of dependence, in the framework of an incompressible medium, of the structure of the reaction zone on hydrodynamic perturbations allows us to exclude it, in general, from examination and to model the flame by a thermal wave ahead of the heated plane x = 0. Following [2], to simplify the internal structure of the flame zone, we replace the continuously changing flow of gas in this region by a constant stream with value of the parameters averaged over its thickness

$$q = \frac{v_3}{v_1} = \frac{\rho_1}{\rho_3} \approx \frac{T_3}{T_1} = \alpha - \frac{1}{e}(\alpha - 1),\ \alpha = \frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} > 1. \tag{2}$$

Leaving aside the change in thickness of the flame zone, we may assume that a small random perturbation of the combustion processes leads to an identical displacement of its front and rear boundaries and may be represented in the form of traveling wave of length $\lambda$

$$\varepsilon(y,t) = C\psi_0 = C\exp(ihy - i\omega t), \quad h = 2\pi/\lambda. \quad (3)$$




Perturbed state of the flame zone

The result is a perturbed state of flow described by the Navier-Stokes equations linearized in the neighborhood of the unperturbed state

$$\frac{\partial v'_{Sx}}{\partial t} + v_S\frac{\partial v'_{Sx}}{\partial x} + \frac{1}{\rho_S}\frac{\partial p'_S}{\partial x} = \nu_S\Delta v'_{Sx},$$

$$\frac{\partial v'_{Sx}}{\partial x} + \frac{\partial v'_{Sy}}{\partial y} = 0,$$

$$\frac{\partial v'_{Sy}}{\partial t} + v_S\frac{\partial v'_{Sy}}{\partial y} + \frac{1}{\rho_S}\frac{\partial p'_S}{\partial y} = \nu_S\Delta v'_{Sy},$$

$$\nu_S = \frac{\eta_S}{\rho_S}, \quad S = 1, 2, 3, \quad (4)$$

where the primes indicate corresponding perturbations of the velocity and pressure components. Solutions of this last system, satisfying the natural requirement of being finite when $|x| \to \infty$, are represented, in conformity with (3), in the following manner:

$$v'_{Sx} = A_S\psi_S + B_S\varphi_S, \quad v'_{Sy} = i(-1)^S(-A_S\psi_S + \gamma_S B_S\varphi_S),$$

$$\frac{p'_S}{\rho_S v_S} = -\left[1 + (-1)^{S-1}\frac{z}{\alpha^{S-1}}\right]A_S\psi_S,$$

$$z = -\frac{i\omega}{hv_1}, \quad \beta_S = 2\frac{h\nu_S}{v_S}, \quad (5)$$

$$\psi_S = \psi_0\exp(-1)^{S-1}hx, \quad \varphi_S = \psi_0\exp(h\gamma_S x),$$

$$\gamma_S = \frac{1}{\beta_S}\left[1 + (-1)^{S-1}\sqrt{1 + 2\beta_S\left(\frac{z}{\alpha^{S-1}} + \frac{\beta_S}{2}\right)}\right]$$

(when S = 3, $\alpha^{S-1}$ should be replaced by q).

The first terms $\psi_S$ are the perturbations of pressure-velocity (the limiting values of the acoustic waves within the framework of an incompressible medium), while the second terms $\varphi_S$ are turbulent perturbations which, in contrast with the ideal case of [2], in the viscous mixture scheme that we have assumed are not only carried by the stream from the flame into the original mixture and inside the flame zone. However, it is not difficult to verify that the perturbation $\varphi_1$ in region 1 may be neglected in comparison with the other perturbations, since it diminishes extremely rapidly with x. In fact, for gases in ordinary conditions [5], $\nu_1 \simeq 1.5\cdot10^{-5}$ cm$^2$/sec, while the normal combustion velocity, $v_1 \sim 1$ m/sec. Hence $k\gamma_1 \simeq v_1/\nu_1 \approx 10^5$ 1/m, so that, for even $x \sim -10^{-4}$ m, which does not exceed the width of the flame zone observed in the experiment, it follows that $\exp(h\gamma_1 x) \sim 10^{-5}$, or that the term $B_1\varphi_1$ may be neglected. To simplify the analysis, we shall also discard the perturbation $B_3\varphi_3$ in the flame zone. Of course, by neglecting in this way the diffusion of turbulence from the chemical reaction zone within the flame, we have lowered somewhat, a priori, the stablizing effect of viscous dissipation within the flame zone, which cannot play an important role in the subsequent investigation since we shall be interested in the very small relative thickness $L/\lambda$ of this zone. After excluding from consideration terms with $B_1$ and $B_3$, the perturbations (5) in regions 1 and 3 coincide with those for the ideal case [2], which permits us to apply the appropriate inverse feedback equation obtained in [2]. The latter which describes the interaction of hydrodynamic perturbations with the internal structure of the flame zone, or, more precisely, expresses the change of normal combustion velocity under the influence of these perturbations, has the form

$$v'_{1x}\Big|_{x=-L} - \frac{\partial\varepsilon}{\partial t} = v_3\int_{t-\tau}^{t}\frac{\partial v'_{3x}}{\partial x}\Bigg|_{x=v_3(t'-t)} dt', \quad L = v_3\tau. \quad (6)$$

For an expression of the perturbations in the internal region of the flame, 3, in terms of those in the original mixture region, 1, we make use of Loitsyanskii's mass change theorem [5]. For this we take the elementary rectangle abcd (see figure) as a control surface, occupying a small region about $x = -L$, and formed by the coordinate lines in such a way that the front boundary AB of the perturbed flame always stays inside it. Then, linearization of the above theorem in the neighborhood of the undisturbed state finally gives $v'_{3x} = qv_{1x}$, i.e., $A_3 = qA_1$.

The mechanical conditions for joining the streams of burnt and unburnt gas through the flame zone may be obtained with the aid of the theorems relating to change of mass and momentum [5]. As a control surface we choose the elementary curvilinear quadrilateral ABCD (shaded in figure) formed by lines AB and CD parallel to the x axis and by small segments of the front and rear boundaries of the perturbed flame zone. The above theorems may be written in the form [5]

$$\int_{(V)}\frac{\partial\rho}{\partial t}dV + \int_{(\sigma)}\rho v_n d\sigma = 0,$$

$$\int_{(V)}\frac{\partial}{\partial t}(\rho\bar{v})dV + \int_{(\sigma)}\rho v_n\bar{v}d\sigma = \int_{(\sigma)}\bar{p}_n d\sigma, \quad (7)$$

where $\sigma$ is the area of the entire control surface, and V is the volume bounded by it. The velocity vector $\bar{v}$, its normal component $v_n$, and the stress $\bar{p}_n$ must be calculated relative to this fixed control surface, i.e., relative to the perturbed state of the flame zone, if we depart, in accordance with our model, from a change in its thickness. Further, we linearize (7) in the neighborhood of the unperturbed state. Then, from the first equation of (7), under the assumption that the medium is incompressible, we obtain

$$\frac{1}{\alpha}\left(v'_{2x}-\frac{\partial\varepsilon}{\partial t}\right)\bigg|_{x=0}=$$
$$=\left(v'_{1x}-\frac{\partial\varepsilon}{\partial t}\right)\bigg|_{x=-L}-\frac{1}{q}\int_{-L}^{0}\frac{\partial v'_{3y}}{\partial y}\,dx. \qquad (8)$$

In the linearization, the second equation of (7) is projected along the normal and the tangent to the flame zone boundary AB(CD). Here the perturbation of the stress $\bar{p}'_n = \bar{p}'_{n\,\mathrm{ideal}} + \bar{p}'_{n\,\mathrm{visc}}$ is composed of the perturbation of the pressure of the ideal medium and the viscous stress, expressed according to well-known formulas [5] in terms of the perturbed velocities. The result is that from (7) we have, for the normal and tangential directions respectively

$$\left[\frac{p'_2}{\rho_2 v_2}+2v'_{2x}-2\,\frac{\nu_2}{v_2}\,\frac{\partial}{\partial x}\left(v'_{2x}-\frac{\partial\varepsilon}{\partial t}\right)\right]_{x=0}=$$
$$=\left[\frac{p'_1}{\rho_1 v_1}+2v'_{1x}-2\,\frac{\nu_1}{v_1}\,\frac{\partial}{\partial x}\left(v'_{1x}-\frac{\partial\varepsilon}{\partial t}\right)\right]_{x=-L}-$$
$$-\int_{-L}^{0}\left\{\frac{1}{v_3}\,\frac{\partial}{\partial t}\left(v'_{3x}-\frac{\partial\varepsilon}{\partial t}\right)+\frac{\partial v'_{3y}}{\partial y}-\right.$$
$$\left.-\frac{\nu_3}{v_3}\,\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(v'_{3x}-\frac{\partial\varepsilon}{\partial t}\right)+\frac{\partial v'_{3y}}{\partial x}\right]\right\}dx,$$
$$\left\{v'_{2y}+v_2\,\frac{\partial\varepsilon}{\partial y}-\frac{\nu_2}{v_2}\left[\frac{\partial}{\partial y}\left(v'_{2x}-\frac{\partial\varepsilon}{\partial t}\right)+\frac{\partial v'_{2y}}{\partial x}\right]\right\}_{x=0}=$$
$$=\left\{v'_{1y}+v_1\,\frac{\partial\varepsilon}{\partial y}-\frac{\nu_1}{v_1}\left[\frac{\partial}{\partial y}\left(v'_{1x}-\frac{\partial\varepsilon}{\partial t}\right)+\right.\right.$$
$$\left.\left.+\frac{\partial v'_{1y}}{\partial x}\right]\right\}_{x=-L}-\int_{-L}^{0}\left\{\frac{1}{v_3}\,\frac{\partial}{\partial t}\left(v'_{3y}+v_3\,\frac{\partial\varepsilon}{\partial y}\right)+\right.$$
$$\left.+\frac{\partial}{\partial y}\left(\frac{p'_3}{\rho_3 v_3}\right)-2\,\frac{\nu_3}{v_3}\,\frac{\partial^2 v'_{3y}}{\partial y^2}\right\}dx. \qquad (9)$$

Substituting the solutions (5) and (3) into the conditions (6), (8), and (9) for joining the flows of burnt and unburnt gas through the flame zone, we come to a linear system of homogeneous equations relating to the constants $A_1$, $A_2$, $B_2$, C, for which the determinant gives an equation for finding the eigenvalue z:

$$F_1F_2-F_3F_4=0,$$
$$F_1=\frac{1}{2}(\gamma_2-1)\,\alpha^m\beta_1-1,$$
$$F_2=-2z\,(z+\alpha+\alpha^{m+1}\beta_1)\exp(\xi)+$$
$$+f\left\{\xi z\left(\frac{z}{q}+1+\beta_1\,\frac{q^m}{2}\right)+(\alpha-1)\left(\frac{z^2}{\alpha}+\right.\right.$$
$$\left.\left.+2z+1\right)+z\beta_1\left[\alpha^m\left(2\alpha-\frac{3}{2}\right)-\frac{1}{2}\right]\right\},$$
$$F_3=\alpha^m\beta_1,$$
$$F_4=\{1+q\,[\exp(\xi)-1]\}\,z+\alpha\exp(\xi)\,z\,(1+\beta_1\alpha^m)-$$
$$-\beta_1 z\,\{1+[\exp(\xi)-1]\,q^{m+1}\}+f\left[(1-\alpha)(1+\right.$$
$$\left.+\beta_1\alpha^m)\,z+1-\alpha-\xi z+\frac{1}{2}\,z\beta_1(1-\alpha^m)\right],$$
$$f=1-\frac{q}{z/q+1}\left[\exp(\xi)-\exp\left(-\frac{z}{q}\,\xi\right)\right],$$
$$\xi=hL=2\pi\,\frac{L}{\lambda}. \qquad (10)$$

From the fact that the Prandtl number is constant for gases, we derive [5] a relation between the thermal diffusivity and the viscosity $\chi_1 \simeq \nu_1$. Hence the viscous and the thermal thicknesses of the normal flame are of the same order, this being expressed, as is well-known [1], in the form of $L \simeq \chi_1/v_1$. Therefore, for the parameter $\beta_1$ which depends on viscosity and appears in (10), we have $\beta \simeq 2\xi$. Experiments with spherical flames have shown [3,4] that instability of normal combustion begins to appear only at very small $\xi$. Therefore, a solution of (10) may be sought in the form of the expansion $z = z_0 + z_1\xi + \ldots$.

Restricting ourselves to a linear approximation with respect to $\xi$, we obtain from (10)

$$z_0=\frac{\alpha}{\alpha+1}\left(\pm\sqrt{\alpha+1-\frac{1}{\alpha}}-1\right),\quad z_1=z_{11}+\frac{\beta_1}{2\xi}\,z_{12},$$
$$\left(\frac{\alpha+1}{\alpha}\,z_0+1\right)z_{11}=-\alpha q\,\frac{\alpha-1}{\alpha+1}\left\{1+\frac{1}{q}\left(1-\frac{1}{2q}\right)+\right.$$
$$\left.+z_0\left[1+\frac{1}{q}-\frac{\alpha+1}{2\alpha q(\alpha-1)}\left(1-\frac{2\alpha}{\alpha+1}\,\frac{1}{q}\right)\right]\right\}, \qquad (11)$$
$$\left(\frac{\alpha+1}{\alpha}z_0+1\right)z_{12}=-\alpha^m(\alpha-1)\left(1-z_0\,\frac{\alpha^m-1}{\alpha-1}\,\frac{1}{\alpha^m}\right).$$

Hence it is seen immediately that for an unstable root $z_0 > 0$ of the zero-order approximation, we always have $z_{11}$, $z_{12} < 0$. The term $z_{12}$ reflects the viscous effect, and (11), regardless of the erroneous conclusion of Einbinder [1], points to the stabilizing influence of the viscous forces, in conformity with the physical nature of this dissipative factor. As far as the other mechanism of energy dissipation, that due to thermal conduction, is concerned, it mav be verified, by a method similar to the foregoing, that its influence is an effect of the order of the square of the Mach number of the main flow, which means that it is negligible in the framework of the supposed incompressibility of the medium.

In making a comparison with experimental observation, one would expect that perturbations of maximum instability should first be realized experimentally, those for which the most rapid growth of amplitude with time has been achieved. Then, from an extremum condition for the single parameter available to us, h: $d(-i\omega)dh = 0$, we find the wavelength of such maximally unstable perturbations of the flame:

$$\xi_m=2\pi L/\lambda_m=-z_0/2z_1. \qquad (12)$$

Hence, putting $\alpha \simeq 12$, and m = 1 for oxygen mixtures, following calculations according to Eqs. (11) and (12), we have the following expected wave dimension for the perturbation that increases the most rapidly with time:

$$\lambda_m/L \simeq 0.7 \cdot 10^3.$$

In [3,4] the results are presented of experiments which were performed on the propagation of spherical flames in ordinary conditions inside a soap bubble for a mixture of 67.5% oxygen and 32.5% acetylene ($\alpha \simeq 12$, according to [6]). Instability of the flames was observed beginning with $r/L \simeq 10^4$.

The upper curve of Fig. 7 of [3] describes the dependence of the width of an inhomogeneity on the flame sphere (this corresponds to $\lambda$ in our notation) on its radius r, while the mean line expresses the amplitude of these perturbations. Taking the data from the graph for the first experimental point, i.e., the point satisfying the least amplitude of perturbation on the sphere of the flame, we have $\Delta x \approx \lambda \leq 4$ mm, $r \approx 25$ mm. Thus, the relationship $r/\lambda \simeq 7$ occurs, and the perturbations observed in the experiments begin to exhibit instability when their wavelength reaches $\lambda/L \simeq 1.4 \cdot 10^3$. Then the amplitude of the inhomogeneities on the perturbed flame sphere is equal to 2 mm. As it decreases, $r/\lambda$ increases. Thus, for 1 mm, $r/\lambda \simeq 11$ already, so that the instability begins to appear with a wavelength of $\lambda/L \simeq 0.9 \cdot 10^3$. The latter agrees well enough with our theoretical value for the wavelength $\lambda_m/L = 0.7 \cdot 10^3$ of the maximally unstable perturbation, if we take into account, in addition, that we lowered the stabilizing effect somewhat by neglecting the diffusion of turbulence inside the flame zone.

In conclusion we note that the validity of the comparison that we have made of a theoretical investigation of a plane flame with an experimental investigation for a spherical flame, is based on the results of Eckhaus [7], who showed, that with $r/L \gg 1$, the curvature of a spherical flame has no influence on the velocity of propagation of combustion.

## NOTATION

p is the pressure; $\rho$ is the density; v is the velocity; T is the temperature; $\eta$ is the viscosity; $\varkappa$ is the thermal diffusivity; $\lambda$ is the wavelength of perturbation; $\omega$ is the unknown eigenvalue; L is the thickness of flame; h is the wave number; $\varepsilon$ is the displacement of flame; $\tau$ is the characteristic combustion time; $\alpha = v_2/v_1$; $\nu$ is the kinematic viscosity; $\beta_1 = 2h\nu_1/v_1$.

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